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Quantum Group Structure of Lie Superalgebra *osp*(1/2)

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We quantize the enveloping Lie superalgebra $U(osp(1/2))$ in the nonstandard simple root system with two odd simple roots. The quantum supergroup structure associated with the nonstandard simple root system is also given.

Quantum groups, quantum vector spaces, and the underlying notion of deformations have substantially enriched the area of mathematics and mathematical physics. The algebraic structure of the quantum group was created by Jimbo and Drinfeld [1, 2]. The one-parameter quantum groups are introduced as one-parameter deformations of the universal enveloping algebra $U(L)$ of an algebra L leading to a noncommutative and noncocommutative Hopf algebra $U_q(L)$, namely the quantum group, Quantum groups can also be considered as nontrivial generalizations of ordinary Lie groups. If L is a Lie algebra, we deal with quantum groups, and if L is a Lie superalgebra, we deal with quantum supergroups [3–6].

Let us consider the Lie superalgebra $osp(1/2)$ whose even part is the Lie algebra $sl(2)$ with generators J_3, $J_\pm$; we call them isospin. The odd generators $V_\pm$ are $sl(2)$ spinors with the following commutation relations:

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_\pm, V_\mp] = -V_\pm$$

$$[J_3, V_\pm] = \pm 1/2 V_\pm, \qquad [V_\pm, V_\pm] = \pm J_\pm$$

$$[J_+, J_-] = 2J_3, \qquad [V_+, V_-] = J_3$$

$$[J_\pm, V_\pm] = 0$$

We will work over a field K with $\mathrm{char}(K) = 0$; we assume that

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$$V_+^2 = f(q)J_+$$
$$V_-^2 = g(q)J_-$$
$$[J_3, V_\pm] = \pm 1/2 V_\pm$$

where $f(q)$ and $g(q)$ are functions of q, and $q \in K^*$ is generic. In a direct way we can prove that our assumptions are consistent with the commutation relations of the Lie superalgebra $osp(1/2)$, that is,

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-$$

Consider also the assumption

$$\{V_+, V_-\} = F(J_3)$$

where $F(J_3)$ is an arbitrary function of J_3. In this case we deduce that

$$[J_+, V_-] = \frac{1}{f(q)}\{F(J_3 - 1/2) - F(J_3)\}V_+ \tag{1}$$

$$[J_-, V_+] = \frac{1}{g(q)}\{F(J_3 + 1/2) - F(J_3)\}V_- \tag{2}$$

$$[J_+, J_-] = \frac{1}{f(q)g(q)}\{\{F(J_3 - 1/2) - F(J_3)\}V_+V_- + \{F(J_3) - F(J_3 + 1/2)\}V_-V_+\} \tag{3}$$

Definition 1. We define the following three types of q-number:

$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}$$

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$$

$$[x]_+ = \frac{q^{x/2} + q^{-x/2}}{q^{1/2} + q^{-1/2}}$$

with $[x] + [x] = [x]_q$, $[x]_q = [x]_{q^{-1}}$, and $[x]_q \to x$ as $q \to 1$.

If we choose $F(J_3) = [J_3]_q$, then we obtain

$$[J_+, V_-] = \frac{-1}{f(q)}[1/2][2J_3 - 1/2]_+ V_+ \tag{4}$$

$$[J_-, V_+] = \frac{1}{g(q)}[1/2][2J_3 + 1/2]_+ V_- \tag{5}$$

$$[J_+, J_-] = \frac{1}{f(q)g(q)} \{-[1/2][2J_3 + 1/2]_+[J_3]_q + (q^{1/4} - q^{-1/4})^2[J_3]_q V_+V_-\} \quad (6)$$

As q tends to 1, Eq. (4) becomes

$$[J_+, V_-] = \frac{-1}{2f(1)} V_+, \qquad \text{but} \quad [J_+, V_-] = -V_+$$

This implies that $f(1) = 1/2$. Also Eq. (5) becomes

$$[J_-, V_+] = \frac{1}{2g(1)} V_-, \qquad \text{but} \quad [J_-, V_+] = -V_- \quad \text{then} \quad g(1) = -1/2$$

We can choose $f(q)$ and $g(q)$ as follows:

$$f(\mathrm{q}) = \frac{1}{q^{1/2} + q^{-1/2}}, \qquad g(q) = \frac{-1}{q^{1/2} + q^{-1/2}}$$

or any other form such that $f(q) \to 1/2$ as $q \to 1$ and $g(q) = -1/2$ as $q \to 1$.

We get the following q-deformed $osp(1/2)$ superalgebra:

$$\{V_+, V_-\} = [J_3]_q \quad (7)$$

$$[J_3, J_\pm] = \pm J_\pm \quad (8)$$

$$[J_3, V_\pm] = \pm 1/2 V_\pm \quad (9)$$

$$[J_\pm, V_\pm] = 0 \quad (10)$$

$$[V_\pm, V_\pm] = \pm J_\pm \quad (11)$$

$$[J_+, V_-] = -(q^{1/2} + q^{-1/2})[1/2][2J_3 - 1/2]_+V_+ \quad (12)$$

$$[J_-, V_+] = -(q^{1/2} + q^{-1/2})[1/2][2J_3 + 1/2]_+V_- \quad (13)$$

$$[J_+, J_-] = -(q^{1/2} + q^{-1/2})^2 \{-[1/2][2J_3 + 1/2]_+[J_3]_q + (q^{1/4} - q^{-1/4})^2[J_3]_q V_+V_-\} \quad (14)$$

Define the comultiplication Δ, the counit ε, and the antipode s for $U_q(osp(1/2))$ as follows:

$$\Delta(q^{\pm J_3}) = q^{\pm J_3} \otimes q^{\pm J_3} \quad (15)$$

$$\Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3 \quad (16)$$

$$\Delta(J_\pm) = J_\pm \otimes q^{J_3} + q^{-J_3} \otimes J_\pm \quad (17)$$

$$\Delta(V_\pm) = V_\pm \otimes q^{J_3} + q^{-J_3} \otimes V_\pm \quad (18)$$

$$\varepsilon(J_\pm) = \varepsilon V_\pm = \varepsilon(J_3) = 0 \tag{19}$$

$$\varepsilon(q^{\pm J_3}) = 1 \tag{20}$$

$$s(J_\pm) = -q^{\pm 1} J_\pm \tag{21}$$

$$s(V_\pm) = -q^{\mp 1} V_\pm \tag{22}$$

$$s(J_3) = -J_3 \tag{23}$$

$$s(q^{\pm J_3}) = q^{\mp J_3} \tag{24}$$

Theorem 2. The q-deformed Lie superalgebra $U_q(osp(1/2))$ for $|q| = 1$ under the comultiplication Δ, the counit ε, and the antipode S defined by (15)–(24) is a noncommutative and noncocommutative Hopf superalgebra, subject to the following constraints:

$$q^{\pm J_3}\, q^{\mp J_3} = 1$$

$$q^{J_3} J_\pm\, q^{-J_3} = q^{\pm 1} J_\pm$$

$$q^{J_3} V_\pm q^{-J_3} = q^{\mp 1} V_\pm$$

Proof. From the definition one can see that the Hopf superalgebra is associative, noncommutative, and noncommutative. First, it is easy to see the coassociativity of Δ [i.e., $(\mathrm{I} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{I}) \circ \Delta$] and that the counit ε is an algebraic homomorphism [i.e., $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$; $a, b \in U_q(osp(1/2))$] with the counit property $(I \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{I}) \circ \Delta = I$.

Now we will prove that Δ is an algebraic homomorphism, i.e., Δ preserves the relations (11)–(13).
Thus,

$$\begin{aligned}
\Delta([J_+, V_-]) &= \frac{q^{-1/2} - 1}{(q - q^{-1})f(q)} \Delta\{q^{J_3} V_+ + q^{1/2} q^{-J_3} V_+\} \\
&= \frac{q^{-1/2} - 1}{(q - q^{-1})f(q)} \{(q^{J_3} \otimes q^{J_3})\,(V_+ \otimes q^{J_3} + q^{-J_3} \otimes V_+) \\
&\quad + q^{1/2}(q^{-J_3} \otimes q^{-J_3})(V_+ \otimes q^{J_3} + q^{-J_3} \otimes V_+)\} \\
&= \frac{q^{-1/2} - 1}{(q - q^{-1})f(q)} \{q^{J_3} V_+ \otimes 1 + 1 \otimes q^{J_3} V_+ \\
&\quad + q^{1/2} q^{-J_3} V_+ \otimes 1 + q^{1/2} 1 \otimes q^{-J_3} V_+\} \\
&= \frac{q^{-1/2} - 1}{(q - q^{-1})f(q)} \{(q^{J_3} V_+ + q^{1/2} q^{-J_3} V_+) \otimes 1 \\
&\quad + 1 \otimes (q^{J_3} V_+ + q^{1/2} q^{-J_3}\, V_+)\}
\end{aligned}$$

$$= [J_+, V_-] \otimes 1 + 1 \otimes [J_+, V_-] \tag{i}$$

Also,

$$\begin{aligned}[\Delta J_+, \Delta V_-] &= [J_+ \otimes q^{J_3} + q^{-J_3} \otimes J_+, V_- \otimes q^{J_3} + q^{-J_3} \otimes V_-] \\ &= [J_+ \otimes q^{J_3}, V_- \otimes q^{J_3}] + [J_+ \otimes q^{J_3}, q^{-J_3} \otimes V_-] \\ &\quad + [q^{-J_3} \otimes J_+, V_- \otimes q^{J_3}] + [q^{-J_3} \otimes J_+, q^{-J_3} \otimes V_-]\end{aligned}$$

but

$$\begin{aligned}[J + \otimes q^{J_3}, q^{-J_3} \otimes V_-] &= (J_+ q^{-J_3} \otimes q^{J_3} V_-) - (q^{-J_3} J_+ \otimes V_- q^{J_3}) \\ &= (q q^{-J_3} J_+ \otimes q V_-\, q^{J_3}) - (q^{-J_3} J_+ \otimes V_- q^{J_3}) = 0\end{aligned}$$

Similarly

$$[q^{-J_3} \otimes J_+, V_- \otimes q^{J_3}] = 0$$

$$\begin{aligned}[J_+ \otimes q^{J_3}, V_- \otimes q^{J_3}] &= J_+ V_- \otimes q^{2J_3} - V_- J_+ \otimes q^{2J_3} \\ &= (J_+ V_- - V_- J_+) \otimes 1 \\ &= [J_+, V_-] \otimes 1\end{aligned}$$

Similarly,

$$[q^{-J_3} \otimes J_+, q^{-J_3} \otimes V_-] = 1 \otimes [J_+, V_-]$$

Then

$$[\Delta J_+, \Delta V_-] = [J_+, V_-] \otimes 1 + 1 \otimes [J_+, J_-] \tag{ii}$$

From (i) and (ii) we get that $\Delta([J_+, V_-]) = [\Delta J_+, \Delta V_-]$. In the same way we can prove that Δ is an algebraic homomorphism for the other relations, and $U_q(osp(1/2))$ is bi-superalgebra.

It remains to prove the equations of the antipode s. The antipode s is a z_2-graded vector space homomorphism:

$$s: \quad U_q(osp(1/2)) \rightarrow U_q(osp(1/2))$$

satisfies $M \circ (I \otimes s) \circ \Delta = M \circ (s \otimes I) \circ \Delta = u \circ \epsilon$; M is the multiplication in $U_q(osp(1/2))$ and u is the unit u: $K \rightarrow U_q(osp(1/2))$.

The antipode s can be extended to be an antialgebra, i.e.,

$$s(ab) = (-1)^{|a||b|} s(b) s(a)$$

for any $a, b \in U_q(osp(1/2))$, where $|a|$ is the grade of the homogeneous element a such that it satisfies the following:

(i) $s \circ u = u$.

(ii) $\varepsilon \circ s = \varepsilon$.

(iii) $T \circ (s \otimes s) \circ \Delta = \Delta \circ s$, where T is the twisting map defined by $T(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

(iv) $s^2 = I$.

It is clear that s satisfies (i) and (ii). For (iii) we will check the relation for the elements $q^{\pm J_3}$:

$$\begin{aligned} T \circ (s \otimes s) \circ \Delta\, (q^{\pm J_3}) &= T \circ (s \otimes s)(q^{\pm J_3} \otimes q^{\pm J_3}) \\ &= T(q^{\mp J_3} \otimes q^{\mp J_3}) = (q^{\mp J_3} \otimes q^{\mp J_3}) \\ &= \Delta(q^{\mp J_3}) = \Delta(s(q^{\pm J_3})) = \Delta \circ s(q^{\pm J_3}) \end{aligned}$$

Similarly for the elements $J_\pm$ and $V_\pm$. Let us now prove that

$$s^2([J_+, V_-]) = [J_+, V_-]$$

Thus,

$$\begin{aligned} s^2\,([J_+, V_-]) &= s\left(\frac{q^{-1/2} - 1}{f(q)(q - q^{-1})} \{-q^{-1}\, V_+ q^{-J_3} - q^{-1/2}\, q^{-1}\, V_+ q^{J_3}\}\right) \\ &= -\frac{q^{-1/2} - 1}{f(q)(q - q^{-1})} S\{q^{-J_3} V_+ + q^{1/2} q^{J_3} V_+\} \\ &= -\frac{q^{-1/2} - 1}{f(q)(q - q^{-1})} \{-q^{-1} V_+ q^{J_3} - q^{1/2} q^{-1} V_+\, q^{-J_3}\} \\ &= -\frac{q^{-1/2} - 1}{f(q)(q - q^{-1})} \{q^{J_3} V_+ + q^{1/2} q^{-J_3} V_+\} \\ &= [J_+, V_-] \end{aligned}$$

We can of course prove this property for other elements of the superalgebra $U_q(osp(1/2))$.

Now we will prove that the antipode s obeys

$$M \circ (I \otimes s) \circ \Delta = M \circ (s \otimes I) \circ \Delta = u \circ \varepsilon$$

Thus,

$$\begin{aligned} M \circ (I \otimes s) \circ \Delta(J_\pm) &= M \circ (I \otimes s)(J_\pm \otimes q^{J_3} + q^{-J_3} \otimes J_\pm) \\ &= M(J_\pm \otimes q^{-J_3} - q^{\pm 1} q^{-J_3} \otimes J_\pm) \\ &= J_\pm\, q^{-J_3} - q^{\pm 1} q^{-J_3} J_\pm \\ &= q^{\pm 1} q^{-J_3} J_\pm - q^{\pm 1} q^{-J_3} J_\pm = 0 \end{aligned}$$

$$M \circ (s \otimes I) \circ \Delta(J_\pm) = M \circ (s \otimes I)(J_\pm \otimes q^{J_3} + q^{-J_3} \otimes J_\pm)$$

$$= M(-q^{\pm 1}J_{\pm} \otimes q^{J_3} + q^{J_3} \otimes J_{\pm})$$
$$= -q^{\pm 1}\, J_{\pm}q^{J_3} + q^{J_3}J_{\pm}$$
$$= -q^{\pm 1}J_{\pm}q^{J_3} + q^{\pm 1}J_{\pm}q^{J_3} = 0$$

Similarly for the other elements. Therefore $U_q(osp(1/2))$ is a noncommutative, noncocommutative Hopf superalgebra, that is, $U_q(osp(1/2))$ is a quantum supergroup and the proof is completed.

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